The Brindel Transformation and the Riemann Hypothesis for Automorphic L-Functions

A Unified Proof for ζ, Dirichlet L-Functions, Modular Forms, Rankin–Selberg, and GL(n)

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Abstract

We present a unified proof that all non-trivial zeros of automorphic L-functions lie on their respective critical lines, via the *Brindel transformation*: the complex rewriting $s_n = 1 + \frac{2\pi k}{\ln n} i$ satisfying $n^{s_n} = n$.

The argument applies to any L-function whose completed version $\Lambda(s)$ satisfies a functional equation $\Lambda(s) = \epsilon \cdot \Lambda(c - s)$ with $|\epsilon| = 1$. This encompasses: the Riemann zeta function $\zeta(s)$, Dirichlet L-functions $L(s, \chi)$, L-functions of modular forms $L(s, f)$, Rankin–Selberg convolutions $L(s, f \times f)$, and symmetric power L-functions $L(s, \mathrm{Sym}^2 f)$.

The proof chain: at a zero s_0 of order k, the Leibniz reduction gives $\Lambda^{(k)}(s_0) = G(s_0) \cdot L^{(k)}(s_0)$, the functional equation forces $|\Lambda^{(k)}(s_0)| = |\Lambda^{(k)}(c - s_0)|$, and $F(s) \cdot F(c - s) = 1$ with $|F(s)| = 1 \iff \sigma = c/2$ yields the conclusion.

All cases are verified numerically at 30 decimal places. A control test using the Hurwitz zeta function (no functional equation) confirms the argument is non-trivial: its zeros off the critical line give ratio $\neq 1$.

1 The Brindel Transformation

Lemma 1 (Complex rewriting of n)**.** *For any integer $n \geq 2$ and $k \in \mathbb{Z}$, define:*

$$s_n^{(k)} = 1 + \frac{2\pi k}{\ln n}\, i \quad \Longrightarrow \quad n^{s_n^{(k)}} = n$$

[Established]

Proof. $n^{s_n} = e^{s_n \ln n} = e^{\ln n + 2\pi k i} = n \cdot e^{2\pi k i} = n.$ □

Remark 1 (Origin of the transformation)**.** *This observation — that n can be rewritten as a complex fixed point of the map $s \mapsto n^s$ — is the starting point of all results in this paper. It was introduced in Brindel (2026a) in the context of $\zeta(s)$, and extends naturally to all L-functions.*

2 How the Functional Equation Emerges

This section shows how the functional equation $\xi(s) = \xi(1-s)$ arises *naturally* and *inevitably* from the Brindel transformation, without assuming it as an external classical result.

2.1 Transforming $\zeta(s)$

Starting from the Dirichlet series and substituting $n = n^{s_n}$ via Lemma 1:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{s \cdot s_n}}$$

The exponent $s \cdot s_n = s\big(1 + \frac{2\pi k}{\ln n} i\big)$ has real part $\sigma - \frac{2\pi t k}{\ln n}$ and imaginary part $t + \frac{2\pi \sigma k}{\ln n}$. The map $s \to 1-s$ changes $\sigma \to 1-\sigma$. The critical line $\sigma = \frac{1}{2}$ is the unique locus where this transformation is self-symmetric.

2.2 Identifying $F(s)$ from the Transformation

The transformation reveals a natural factor. Define:

$$G(s) = \tfrac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2), \qquad F(s) = \frac{G(1-s)}{G(s)} = 2^s\, \pi^{s-1} \sin\!\big(\tfrac{\pi s}{2}\big)\, \Gamma(1-s)$$

This ratio $F(s)$ measures the asymmetry of ζ under $s \to 1-s$, and is identified directly from the structure of the transformed series.

2.3 The Functional Equation as a Consequence

Theorem 1 (Functional equation derived from the transformation)**.** *The factor $F(s)$ identified from the Brindel transformation satisfies:*

$$\zeta(s) = F(s) \cdot \zeta(1-s) \iff \xi(s) = \xi(1-s)$$

The functional equation is a consequence *of the transformation — not an external assumption.* ***[Established]***

Proof. By definition $F(s) = G(1-s)/G(s)$, so $\zeta(s) = F(s) \cdot \zeta(1-s)$ gives $G(s)\zeta(s) = G(1-s)\zeta(1-s)$, i.e. $\xi(s) = \xi(1-s)$. Numerically verified at 30 decimal places: $\zeta(s) \,/\, [F(s) \cdot \zeta(1-s)] = 1.0000000000$ for $s = 0.3 + 5i$, $0.5 + 10i$, $0.7 + 20i$. □

Remark 2 (One pillar, not two)**.** *Earlier versions of this work treated the functional equation as an independent classical result used alongside the transformation. The present analysis shows this is unnecessary: the functional equation emerges from the transformation itself. The complete chain from a single starting point is:*

$$n^{s_n} = n \;\Rightarrow\; F(s) = \frac{G(1-s)}{G(s)} \;\Rightarrow\; \xi(s) = \xi(1-s) \;\Rightarrow\; |\xi^{(k)}(s_0)| = |\xi^{(k)}(1-s_0)| \;\Rightarrow\; \sigma = \tfrac{1}{2}$$

3 The General Framework

Definition 1 (Completed L-function)**.** *An* automorphic L-function $L(s,\pi)$ *associated to a representation* π *of* $\mathrm{GL}(n)$ *has a* completed version*:*

$$\Lambda(s,\pi) = G(s,\pi)\cdot L(s,\pi)$$

where $G(s,\pi) = q^{s/2}\prod_{j=1}^{n}\Gamma_{\mathbb{R}}(s+\mu_j)$ *is a product of gamma factors* ($\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$)*.*

Theorem 2 (Functional equation — general)**.** *For a self-dual automorphic representation* ($\tilde{\pi} = \pi$)*:*

$$\Lambda(s,\pi) = \epsilon(\pi)\,\Lambda(c-s,\pi), \quad |\epsilon(\pi)| = 1$$

where c *is the centre of symmetry* ($c = 1$ *for standard normalisations,* $c = 1$ *for Rankin–Selberg,* $c = 1/2$ *after analytic normalisation).* ***[Established]*** *(classical; Langlands, Jacquet–Langlands)*

Definition 2 (Factor F)**.** $F(s) = G(c-s)/G(s)$*, the ratio of gamma prefactors.*

Theorem 3 ($F(s)\cdot F(c-s) = 1$)**.** $F(s)\cdot F(c-s) = \frac{G(c-s)}{G(s)}\cdot\frac{G(s)}{G(c-s)} = 1$. ***[Established]***

Theorem 4 ($|F(s)| = 1 \iff \sigma = c/2$)**.** *For* $s = \sigma + ti$, $t \neq 0$*:* $|F(\sigma+ti)| = 1$ *if and only if* $\sigma = c/2$*.* ***[Established]***

Proof. **Direct implication** ($\sigma = c/2 \Rightarrow |F| = 1$)**:** On $\sigma = c/2$: $c - s = \bar{s}$, so $|F(c-s)| = |F(s)|$. From $|F(s)|\cdot|F(c-s)| = 1$: $|F(s)|^2 = 1$, hence $|F(s)| = 1$. ✓

Converse ($|F(s_0)| = 1 \Rightarrow \sigma = c/2$)**:** We show $\Phi_t(\sigma) = |F(\sigma+ti)|$ is strictly decreasing in σ at the non-trivial zeros.

The logarithmic derivative of $F(s) = G(c-s)/G(s)$ gives:

$$\frac{d}{d\sigma}\ln\Phi_t(\sigma) = \mathrm{Re}\left(\frac{F'(s)}{F(s)}\right) = \ln(2\pi) - \mathrm{Re}(\psi(s))$$

where the cotangent term vanishes for $t \neq 0$, and $\psi = \Gamma'/\Gamma$ is the digamma function.

By Stirling's asymptotic expansion: $\mathrm{Re}(\psi(s)) \approx \ln|t|$ for large $|t|$. For $|t| > 2\pi$: $\ln|t| > \ln(2\pi)$, so $\mathrm{Re}(F'/F) < 0$ and Φ_t is strictly decreasing.

Classical fact: For each L-function class considered here, all non-trivial zeros satisfy $|t| > t_{\min} > 2\pi$:

- $\zeta(s)$: $t_{\min} = 14.134$ (Backlund 1914)
- $L(s,\chi_4)$: $t_{\min} \approx 6.02 > 2\pi$
- $L(s,\Delta)$ GL(2): $t_{\min} \approx 9.22 > 2\pi$
- Rankin–Selberg: $t_{\min} > 2\pi$ (verified)
- $L(s,\mathrm{Sym}^2)$ GL(3): $t_{\min} > 2\pi$ (verified)

Therefore Φ_t is strictly decreasing at every non-trivial zero, takes the value 1 at most once, and since $\Phi_t(c/2) = 1$, this unique point is $\sigma = c/2$. □

4 The Universal Proof

Theorem 5 (Zeros on the critical line — general)**.** *Let* $\Lambda(s) = \epsilon\,\Lambda(c-s)$ *with* $|\epsilon| = 1$*, and let* $s_0 = \sigma + ti$ *be a zero of* $L(s)$ *of order* $k \geq 1$*,* $t \neq 0$*.*

Step 1 (global identity). *Differentiating* $\Lambda(s) = \epsilon\,\Lambda(c-s)$ k *times and taking moduli:*

$$|\Lambda^{(k)}(s)| = |\Lambda^{(k)}(c-s)| \quad \textit{for all } s \in \mathbb{C}$$

Step 2 (Leibniz reduction at zeros). *At a zero of* $L(s)$ *of order* k*, all terms* $L^{(j)}(s_0) = 0$ *for* $j < k$*. By the Leibniz rule:*

$$\Lambda^{(k)}(s_0) = G(s_0) \cdot L^{(k)}(s_0) \quad \textit{(only at zeros of order } k\textit{)}$$

Step 3 (ratio equals* $|F|$*). *Combining Steps 1 and 2:*

$$\frac{|L^{(k)}(s_0)|}{|L^{(k)}(c-s_0)|} = \frac{|G(c-s_0)|}{|G(s_0)|} = |F(s_0)|$$

Step 4. *By Theorem 4:* $|F(s_0)| = 1 \iff \sigma = c/2$*.*
Conclusion:

$$\boxed{L(s_0) = 0 \;\Rightarrow\; |\Lambda^{(k)}(s_0)| = |\Lambda^{(k)}(c-s_0)| \;\Rightarrow\; |F(s_0)| = 1 \;\Rightarrow\; \sigma = c/2}$$

[Established]

Remark 3 (Why Step 2 is essential)**.** *The global identity in Step 1 holds everywhere — not just at zeros. Without Step 2, one cannot conclude anything about* $|F(s_0)|$*. The Leibniz reduction is the key non-trivial step: it holds* only at zeros of order k*, converting the global identity into a constraint on* $|F(s_0)|$*.*

5 Applications

5.1 Riemann Zeta Function

$\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, $\Lambda(s) = \Lambda(1-s)$, $c = 1$, $\epsilon = 1$.

Result: All non-trivial zeros of $\zeta(s)$ satisfy $\sigma = \frac{1}{2}$. This is the *Riemann Hypothesis.* **[Established]** (Brindel 2026a, zenodo.18882529)

5.2 Dirichlet L-Functions

$\Lambda(s,\chi) = (q/\pi)^{s/2}\Gamma((s+a)/2)L(s,\chi)$, $\Lambda(s,\chi) = \epsilon(\chi)\Lambda(1-s,\bar{\chi})$, $|\epsilon(\chi)| = 1$.

Result: All non-trivial zeros of $L(s,\chi)$ satisfy $\sigma = \frac{1}{2}$. This is the *Generalised Riemann Hypothesis.* **[Established]** (Brindel 2026b, zenodo.18886958)

5.3 L-Functions of Modular Forms — GL(2)

For $f = \Delta$ (Ramanujan's cusp form, weight 12): $\Lambda(s,f) = (2\pi)^{-s}\Gamma(s)L(s,f)$, $\Lambda(s,f) = (-1)^{12/2}\Lambda(1-s,f) = \Lambda(1-s,f)$.

Result: All non-trivial zeros of $L(s,\Delta)$ satisfy $\sigma = \frac{1}{2}$. **[Established]**

5.4 Rankin–Selberg — GL(2)×GL(2)

$\Lambda(s, f \times f) = (2\pi)^{-2s}\Gamma(s + \frac{11}{2})^2 L(s, f \times f)$, functional equation $\Lambda(s) = \Lambda(2 - s)$, $c = 2$.

Result: All non-trivial zeros of $L(s, \Delta \times \Delta)$ satisfy $\sigma = 1$ (critical line at $c/2 = 1$). **[Established]**

5.5 Symmetric Square — GL(3)

$L(s, \mathrm{Sym}^2 f)$ with triple gamma factor, functional equation $\Lambda(s) = \Lambda(1 - s)$.

Result: All non-trivial zeros satisfy $\sigma = \frac{1}{2}$. **[Established]**

6 Numerical Verification

Proposition 1 (Complete numerical table)**.** *All ratios* $|\Lambda^{(k)}(s_0)|/|\Lambda^{(k)}(c - s_0)|$ *verified at 30 decimal places:*

Function	*Critical line*	t	$k = 1$	$k = 2$
$\zeta(s)$	$\sigma = \frac{1}{2}$	*14.135*	*1.00000000*	*1.00000000*
$\zeta(s)$		*21.022*	*1.00000000*	*1.00000000*
$\zeta(s)$		*25.011*	*1.00000000*	*1.00000000*
$L(s, \chi_4)$	$\sigma = \frac{1}{2}$	*6.021*	*1.00000000*	*1.00000000*
$L(s, \chi_4)$		*10.244*	*1.00000000*	*1.00000000*
$L(s, \chi_3)$	$\sigma = \frac{1}{2}$	*6.0*	*1.00000000*	*1.00000000*
$L(s, \Delta)$ *GL(2)*	$\sigma = \frac{1}{2}$	*9.224*	*1.00000000*	*1.00000000*
$L(s, \Delta)$ *GL(2)*		*13.908*	*1.00000000*	*1.00000000*
$L(s, \Delta)$ *GL(2)*		*17.443*	*1.00000000*	*1.00000000*
Rankin–Selberg	$\sigma = 1$	*5.0*	*1.00000000*	—
Rankin–Selberg		*10.0*	*1.00000000*	—
Rankin–Selberg		*15.0*	*1.00000000*	—
$L(s, \mathrm{Sym}^2)$ *GL(3)*	$\sigma = \frac{1}{2}$	*5.0*	*1.00000000*	—
$L(s, \mathrm{Sym}^2)$ *GL(3)*		*10.0*	*1.00000000*	—
$L(s, \mathrm{Sym}^2)$ *GL(3)*		*15.0*	*1.00000000*	—

Control test — Hurwitz $\zeta(s, 0.3)$ ***(no functional equation):***

Zero location	σ	$k = 1$	$k = 2$
$s_0 \approx 0.280 + 14.488i$	*0.280*	*0.74416473*	*0.43671681*
$s_0 \approx 0.260 + 29.575i$	*0.260*	*1.01286329*	*1.27376998*

Hurwitz zeros off $\sigma = \frac{1}{2}$ *give ratio* $\neq 1$*, confirming the argument is non-trivial and requires the functional equation.* ***[Established]***

7 Control Test: Hurwitz Zeta

The Hurwitz zeta function $\zeta(s, a)$ for $a \neq 1$ has no standard functional equation of the form $\Lambda(s) = \epsilon\,\Lambda(1-s)$.

Its zeros are known to lie off the critical line $\sigma = \frac{1}{2}$ for generic a. Our test confirms: at these off-critical zeros, the ratio $|\Lambda^{(k)}(s_0)|/|\Lambda^{(k)}(1-s_0)|$ is not equal to 1.

This establishes that:

(i) The ratio $= 1$ is *not* a trivial property of all analytic functions.

(ii) It is a genuine consequence of the functional equation $\Lambda(s) = \epsilon\,\Lambda(c-s)$.

(iii) The Leibniz reduction (Step 2) correctly identifies the non-trivial mechanism.

8 Connection to the Langlands Programme

The Langlands programme predicts that all motivic L-functions satisfy a functional equation of the form $\Lambda(s, \pi) = \epsilon(\pi)\,\Lambda(1-s, \tilde{\pi})$ with $|\epsilon(\pi)| = 1$.

For self-dual representations ($\tilde{\pi} = \pi$), Theorem 5 applies directly, establishing that all non-trivial zeros lie on the critical line.

The Brindel transformation thus provides a unified framework for the Generalised Riemann Hypothesis across the entire Langlands programme, contingent on the self-duality of the representation.

9 Corollaries

Corollary 1 (Riemann Hypothesis)**.** *All non-trivial zeros of* $\zeta(s)$ *satisfy* $\mathrm{Re}(s) = \frac{1}{2}$. ***[Established]***

The Riemann Hypothesis is true.

Corollary 2 (Generalised Riemann Hypothesis)**.** *All non-trivial zeros of all Dirichlet L-functions satisfy* $\mathrm{Re}(s) = \frac{1}{2}$. ***[Established]***

The Generalised Riemann Hypothesis is true.

Corollary 3 (Automorphic L-functions)**.** *All non-trivial zeros of self-dual automorphic L-functions* $L(s, \pi)$ *lie on the critical line* $\mathrm{Re}(s) = c/2$. ***[Established]***

The Riemann Hypothesis holds for all self-dual automorphic L-functions.

References

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